Int. J. Heat Mass Transfer. Vol. 27, No. 10, pp. 1845–1854, 1984
Printed in Great Britain

0017-9310/84 $3.00+0.00

Propagation and reflection of thermal waves in a finite medium

M. N. ÖZIŞIK
Mechanical and Aerospace Engineering Department, North Carolina State University, Raleigh, NC 27650, U.S.A.

and

BRIAN VICK
Mechanical Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.

(*Received* 19 *July* 1983 *and in revised form* 9 *January* 1984)

Abstract—For situations involving extremely short times following the start of transients or for very low temperatures near absolute zero, the classical diffusion theory of heat conduction breaks down since the wave nature of thermal energy transport becomes dominant. In this work, analytical solutions are developed for the hyperbolic heat conduction equation describing the wave nature of thermal energy transport in a finite slab with insulated boundaries subjected to a volumetric energy source in the medium. The exact analytical solutions developed for the temperature field and heat flux show that the release of a concentrated pulse of energy gives rise to a severe thermal wave front which travels through the medium at a finite propagation speed, dissipating energy in its wake and reflecting from the insulated surfaces.

INTRODUCTION

IN THE classical theory of heat conduction, heat flux is postulated to be directly proportional to temperature gradient in the form

$$q = -k\frac{\partial T}{\partial x}. \tag{1}$$

When this flux law is combined with the following energy conservation equation

$$-\frac{\partial q}{\partial x}+g(x,t) = \rho C_p\frac{\partial T}{\partial t} \tag{2}$$

heat conduction becomes a diffusion phenomenon which predicts that a thermal disturbance at any point in a medium will be instantaneously felt at every other point in the medium. Despite this physically unrealistic notion of instantaneous heat diffusion with infinite propagation speed, heat flow predictions based on flux law, equation (1), and conservation law, equation (2), are quite accurate for most situations of common experience.

However, in situations dealing with transient heat flow in extremely short periods of time or at very low temperatures approaching absolute zero, this classical heat diffusion theory breaks down, because the wave nature of heat propagation becomes dominant. Peshkov [1] was the earliest investigator to detect experimentally the existence of thermal waves using superfluid liquid helium near absolute zero. Since then, the wave nature of heat propagation has been the subject of numerous investigations [2–13].

In order to accommodate the finite propagation speed and to account for the presence of the observed thermal waves, a more precise heat flux law needs to be postulated, since it is desired to retain the basic notion of energy conservation as described by equation (2). Vernotte [2] in 1958 suggested just such a modified heat flux law in the form

$$\tau\frac{\partial q}{\partial t}+q = -k\frac{\partial T}{\partial x}. \tag{3}$$

When a flux law of the form given by equation (3) is used in conjunction with conservation law, equation (2), a hyperbolic equation governing heat conduction results, as opposed to the classical parabolic equation obtained by using flux law, equation (1). Equation (3) is actually a truncated form of a more general relation originally derived by Maxwell [14] from kinetic theory considerations.

In equation (3), τ represents a relaxation time or build-up period for the commencement of heat flow after a temperature gradient has been imposed on the medium. That is, heat flow does not start instantaneously but grows gradually with a relaxation time τ after the application of a temperature gradient. Similarly, the heat flow does not cease immediately, but dies out gradually after a temperature gradient is removed. Formally, the situation $\tau \to 0$ leads to instantaneous diffusion at infinite propagation velocity which coincides with the classical diffusion theory.

When τ is relatively large, as in the case of temperatures near absolute zero or for substances with

NOMENCLATURE

C	propagation speed
C_p	specific heat
g	volumetric energy source
g_0	quantity defined by equation (9)
$G(\eta, \xi \mid \eta_0, \xi_0)$	Green's function
$\bar{G}_m(\xi)$	transform of Green's function
L	region length
$N(\lambda_m)$	mth normalization integral
$q(x, t)$	heat flux
$Q(\eta, \xi)$	dimensionless heat flux
$S(\eta, \xi)$	dimensionless energy source
t	time variable
$T(x, t)$	temperature
T_0	initial temperature
x	space variable.

Greek symbols

α	thermal diffusivity
δ	delta function
$\Delta\eta$	pulse width
η	dimensionless space variable
η_l	dimensionless region length
$\theta(\eta, \xi)$	dimensionless temperature
λ_m	mth eigenvalue
ξ	dimensionless time variable
ρ	density
τ	relaxation time.

an extremely ordered internal structure, thermal disturbances appear to propagate as waves at finite speeds as observed experimentally in refs. [1, 4]. The observation in ref. [4] is particularly interesting. It shows an oscilloscope trace of a pulsed energy source propagating at constant speed in superfluid helium at about 1 K. Superfluid liquid helium can be obtained in extremely pure form since it remains as a liquid all the way to absolute zero, allowing all chemical impurities to be frozen out, and thus provides an excellent medium in which to observe thermal waves.

Theoretical predictions are available in the literature for some specific cases. The wave front resulting from a step change in temperature at the boundary of a medium was demonstrated in refs. [10, 11], while a numerical solution using the method of characteristics was applied in ref. [12] for the case of a step change in the heat flux at the boundary surface of a semi-infinite medium. Very recently, the present authors [13] presented theoretical predictions for the propagation of a single heat pulse in a semi-infinite medium in a manner demonstrated experimentally in ref. [4].

With the advent of laser technology, the use of laser pulses of extremely short duration has found numerous applications for purposes such as the annealing of semiconductors and the surface heating and melting of metals [15–22]. The basic nature of the thermal energy transport immediately after the application of the pulse and the resulting sustained temperature at the surface of the medium have been of interest in such situations. Here, the wave nature of the thermal energy propagation can play an important role in establishing the resulting temperature in the medium.

In exothermic catalytic reactions maximum temperatures in the crystals can occur in extremely short times of the order of 10^{-13} s, hence the treatment of heat conduction as a wave phenomenon becomes of interest [23–27].

When a region of finite thickness with negligible heat loss from the boundary surfaces is considered, the analysis of the propagation of a thermal disturbance due to the release of energy in the medium becomes an intricate matter, since the released energy travels as a wave while dissipating its energy and reflecting off the insulated boundaries. There appears to be no work available in the literature addressing such controversial phenomena. Thus, the objective of the present investigation is to develop solutions for the temperature and heat flux as predicted by the hyperbolic heat equation in a region of finite thickness subjected to a volumetric energy source. The interesting case of the propagation and reflection of a concentrated thermal pulse will be examined in detail.

ANALYSIS

The present investigation concerns a finite region with insulated boundaries where one-dimensional heat conduction and constant thermal properties are considered to prevail. The region is initially in equilibrium at temperature T_0 and the analysis will focus on the propagation of a thermal disturbance initiated by some distributed energy source, $g(x, t)$. The situation is illustrated schematically in Fig. 1.

Governing equations

The propagation of thermal energy is governed by conservation law, equation (2), and flux law, equation (3). When the heat flux is eliminated between these relations, the following hyperbolic heat conduction equation is obtained

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{1}{k}\left[g(x,t)+\frac{\alpha}{C^2}\frac{\partial g}{\partial t}\right],$$
$$\text{in } 0<x<L, \quad t>0. \quad (4)$$

Here the propagation speed, C, is related to the thermal diffusivity and relaxation time by

$$C^2=\frac{\alpha}{\tau}. \quad (5)$$

Formally, the limiting case of zero relaxation time leads to infinite propagation speed and hence to the classical diffusive behavior. We note that the hyperbolic heat

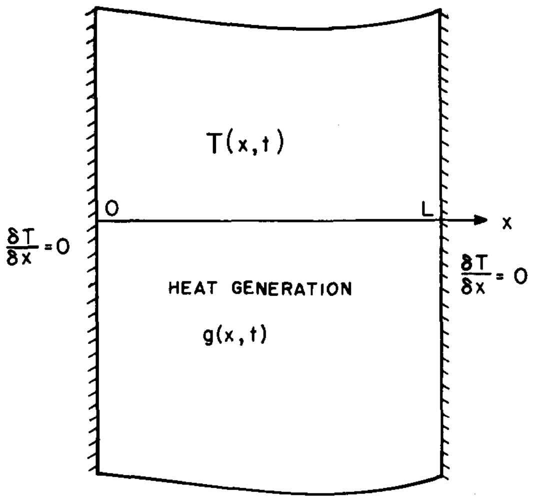

FIG. 1. Geometry and coordinates.

conduction equation with generation includes an additional term involving the time derivative of the generation function, a result also due to the finite propagation speed.

The foregoing hyperbolic heat conduction equation (4) is now considered subject to the boundary conditions

$$\frac{\partial T}{\partial x} = 0, \quad x = 0 \tag{6a}$$

$$\frac{\partial T}{\partial x} = 0, \quad x = L \tag{6b}$$

representing insulated boundaries and by the initial conditions

$$T = T_0, \quad t = 0 \tag{7a}$$

$$\frac{\partial T}{\partial t} = 0, \quad t = 0 \tag{7b}$$

representing thermal equilibrium at a temperature T_0 at time $t = 0$. We point out that even though the heat flux law, equation (3), includes a relaxation or start-up time for heat flow, a temperature gradient is still the motivating force for the transport of thermal energy; thus the boundary conditions, equations (6a) and (6b), indeed represent insulated boundaries.

For convenience in the subsequent analysis, the following dimensionless quantities are introduced

$$\eta = \frac{1}{2}\frac{Cx}{\alpha}, \quad \eta_l = \frac{1}{2}\frac{CL}{\alpha} \tag{8a}$$

$$\xi = \frac{1}{2}\frac{C^2 t}{\alpha} \tag{8b}$$

$$\theta(\eta, \xi) = \frac{T(x,t) - T_0}{g_0 C/k} \tag{8c}$$

$$Q(\eta, \xi) = \frac{q(x,t)}{g_0 C^2/\alpha} \tag{8d}$$

$$S(\eta, \xi) = \frac{g(x,t)}{g_0 C^3/4\alpha^2} \tag{8e}$$

where we utilized the quantity

$$g_0 = \int_{t=0}^{\infty} \int_{x=0}^{L} g(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

which represents the total energy released per unit area normal to the x-axis over the entire region over all time. In order for dimensionless quantities, equations (8c)–(8e), to be meaningful, we restrict the total amount of energy released to be finite ($g_0 < \infty$).

When dimensionless quantities, equations (8a)–(8e), are introduced into equations (4), (6a), (6b), (7a), and (7b), we obtain the following system of equations governing the dimensionless temperature field

$$\frac{\partial^2 \theta}{\partial \xi^2} + 2\frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \eta^2} + \left[S(\eta, \xi) + \frac{1}{2}\frac{\partial S}{\partial \xi}\right], \quad \text{in } 0 < \eta < \eta_l, \quad \xi > 0 \tag{10a}$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \eta = 0 \tag{10b}$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \eta = \eta_l \tag{10c}$$

$$\theta = 0, \quad \xi = 0 \tag{10d}$$

$$\frac{\partial \theta}{\partial \xi} = 0, \quad \xi = 0. \tag{10e}$$

We now describe first the solution to the governing system of equations (10a)–(10e) subject to an arbitrary dimensionless heat generation, $S(\eta, \xi)$, and then apply the results to the interesting special case of a pulsed heat source.

Arbitrary energy source

The analytical solution to system (10) is expressed in terms of a Green's function in the form [28]

$$\theta(\eta, \xi) = \int_{\eta_0=0}^{\eta_l} \int_{\xi_0=0}^{\xi} G(\eta, \xi | \eta_0, \xi_0) \cdot \left[S(\eta_0, \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \xi_0}\right] \mathrm{d}\xi_0\,\mathrm{d}\eta_0. \tag{11}$$

Here $G(\eta, \xi | \eta_0, \xi_0)$ is the appropriate Green's function for the problem which is the solution to governing system (10) with the arbitrary non-homogeneous contribution, $S(\eta, \xi) + 1/2(\partial S/\partial \xi)$, replaced by the unit impulse function $\delta(\eta - \eta_0)\delta(\xi - \xi_0)$. The Green's function represents the fundamental solution to the problem under consideration from which any arbitrary solution can be constructed in the manner described by relation (11).

The key ingredient in the analytical development is the determination of the Green's function itself, which is governed by the following system of equations

$$\frac{\partial^2 G}{\partial \xi^2} + 2\frac{\partial G}{\partial \xi} = \frac{\partial^2 G}{\partial \eta^2} + \delta(\eta - \eta_0)\delta(\xi - \xi_0), \quad \text{in } 0 < \eta < \eta_l, \quad \xi > \xi_0 \tag{12a}$$

$$\frac{\partial G}{\partial \eta} = 0, \quad \eta = 0 \tag{12b}$$

$$\frac{\partial G}{\partial \eta} = 0, \quad \eta = \eta_l \tag{12c}$$

$$G = 0, \quad \xi < \xi_0 \tag{12d}$$

$$\frac{\partial G}{\partial \xi} = 0, \quad \xi < \xi_0. \tag{12e}$$

The initial conditions, equations (12d) and (12e), are based on the causality principle, which states that there can be no effect experienced at times prior to the cause.

The finite integral transform technique is now used to solve system (12). The integral transform pair is defined as [29]

Transform:

$$\bar{G}_m(\xi) = \int_{\eta=0}^{\eta_l} G(\eta, \xi|\eta_0, \xi_0)\cos(\lambda_m \eta)\,\mathrm{d}\eta \tag{13a}$$

Inversion:

$$G(\eta, \xi|\eta_0, \xi_0) = \sum_{m=0}^{\infty} \frac{\bar{G}_m(\xi)}{N(\lambda_m)}\cos(\lambda_m \eta) \tag{13b}$$

where $N(\lambda_m)$ is the normalization integral given by

$$N(\lambda_m) = \begin{cases} \eta_l & m = 0 \\ \frac{1}{2}\eta_l, & m = 1, 2, 3 \ldots \end{cases} \tag{14a}$$

and the λ_m's are the allowable eigenvalues expressed as

$$\lambda_m = \frac{m\pi}{\eta_l}. \tag{14b}$$

In order to determine the transforms, $\bar{G}_m(\xi)$, we operate on equations (12a), (12d), and (12e) with

$$\int_{\eta=0}^{\eta_l} \cos(\lambda_m \eta)\,\mathrm{d}\eta$$

and utilize equations (12b) and (12c) to obtain the following ordinary differential equation and initial conditions

$$\frac{\mathrm{d}^2\bar{G}_m}{\mathrm{d}\xi^2} + 2\frac{\mathrm{d}\bar{G}_m}{\mathrm{d}\xi} + \lambda_m^2 \bar{G}_m(\xi) = \cos(\lambda_m \eta_0)\delta(\xi - \xi_0) \tag{15a}$$

$$\bar{G}_m = 0, \quad \xi < \xi_0 \tag{15b}$$

$$\frac{\mathrm{d}\bar{G}_m}{\mathrm{d}\xi} = 0, \quad \xi < \xi_0. \tag{15c}$$

By a rather lengthy but straightforward series of manipulations, the solution to the system of equations (15a)–(15c) is expressed as

$$\bar{G}_m(\xi) = \mathrm{e}^{-(\xi-\xi_0)}\cos(\lambda_m \eta_0) \times \frac{\sin[(\xi - \xi_0)\sqrt{(\lambda_m^2 - 1)}]}{\sqrt{(\lambda_m^2 - 1)}}, \quad \xi > \xi_0. \tag{16}$$

Substitution of transforms $\bar{G}_m(\xi)$ into inversion formula (13b) immediately gives the desired Green's function as

$$G(\eta, \xi|\eta_0, \xi_0) = \frac{1}{\eta_l}\mathrm{e}^{-(\xi-\xi_0)}\sinh(\xi - \xi_0) + \frac{2}{\eta_l}\mathrm{e}^{-(\xi-\xi_0)}\sum_{m=1}^{\infty}\cos(\lambda_m \eta)\cos(\lambda_m \eta_0) \times \frac{\sin[(\xi - \xi_0)\sqrt{(\lambda_m^2 - 1)}]}{\sqrt{(\lambda_m^2 - 1)}}, \quad \xi > \xi_0. \tag{17}$$

This fundamental solution is now introduced into equation (11) to obtain the temperature distribution as

$$\begin{aligned}\theta(\eta, \xi) = {} & \frac{1}{\eta_l}\int_{\eta_0=0}^{\eta_l}\int_{\xi_0=0}^{\xi} \mathrm{e}^{-(\xi-\xi_0)}\sinh(\xi - \xi_0) \cdot \left[S(\eta_0, \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \xi_0}\right]\mathrm{d}\xi_0\,\mathrm{d}\eta_0 \\ & + \frac{2}{\eta_l}\int_{\eta_0=0}^{\eta_l}\int_{\xi_0=0}^{\xi}\mathrm{e}^{-(\xi-\xi_0)} \times \sum_{m=1}^{\infty}\cos(\lambda_m \eta)\cos(\lambda_m \eta_0) \times \frac{\sin[(\xi - \xi_0)\sqrt{(\lambda_m^2 - 1)}]}{\sqrt{(\lambda_m^2 - 1)}} \\ & \times \left[S(\eta_0, \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \xi_0}\right]\mathrm{d}\xi_0\,\mathrm{d}\eta_0.\end{aligned} \tag{18}$$

The first term on the RHS of equation (18) represents the steady-state portion of the solution that remains after the transients have died out, since any energy released within the insulated region cannot escape, but will merely distribute evenly over the entire region after sufficient time. The temperature distribution resulting from any specified heat source, $S(\eta, \xi)$, is now available using the general solution, equation (18).

Pulsed energy source

We now consider a pulsed energy source released instantaneously at time $t = 0$, in a region $\Delta x \leqslant L$ adjacent to the boundary surface at $x = 0$ with a total strength or energy content per area normal to the x-axis of g_0. Such an energy source could model, for example, the application of a strong laser pulse at the boundary of an absorbing medium encountered in the annealing of semiconductors [15–22].

Such a pulsed energy source can be represented mathematically as

$$g(x, t) = \begin{cases} \dfrac{g_0}{\Delta x}\delta(t), & 0 \leqslant x \leqslant \Delta x \\ 0, & \Delta x < x \leqslant L \end{cases} \tag{19}$$

where $\delta(t)$ is the dirac delta function. Using dimensionless quantities (8a)–(8e), energy source (19) takes the form

$$S(\eta, \xi) = \begin{cases} \dfrac{1}{\Delta \eta}\delta(\xi), & 0 \leqslant \eta \leqslant \Delta \eta \\ 0, & \Delta \eta < \eta \leqslant \eta_l. \end{cases} \tag{20}$$

The temperature distribution is directly available by substituting equation (20) into the general solution, equation (18), and performing the indicated operations to obtain

$$\theta(\eta,\xi) = \frac{1}{2\eta_l} + \frac{1}{\eta_l}\,e^{-\xi}\sum_{m=1}^{\infty}\cos(\lambda_m\eta) \times \frac{\sin(\lambda_m\Delta\eta)}{\lambda_m\Delta\eta}\left[\frac{\sin(\xi\sqrt{(\lambda_m^2-1)})}{\sqrt{(\lambda_m^2-1)}} + \cos(\xi\sqrt{(\lambda_m^2-1)})\right]. \quad (21)$$

The corresponding heat flux is determined by integrating the flux law, equation (3), and expressing the results in dimensionless form as

$$Q(\eta,\xi) = -e^{-2\xi}\int_{\xi'=0}^{\xi} e^{2\xi'}\frac{\partial\theta}{\partial\eta}\,d\xi'. \quad (22)$$

When the temperature distribution, equation (21), is substituted into expression (22), the desired heat flux becomes

$$Q(\eta,\xi) = \frac{1}{\eta_l}\,e^{-\xi}\sum_{m=1}^{\infty}\sin(\lambda_m\eta) \times \frac{\sin(\lambda_m\Delta\eta)}{\Delta\eta}\,\frac{\sin(\xi\sqrt{(\lambda_m^2-1)})}{\sqrt{(\lambda_m^2-1)}}. \quad (23)$$

Before presenting representative results, we examine some interesting limiting cases. First, we consider the limit as the pulsed energy source becomes infinitely concentrated by taking the limit as $\Delta\eta \to 0$ in equations (21) and (23) to obtain

$$\lim_{\Delta\eta\to 0}[\theta(\eta,\xi)] = \frac{1}{2\eta_l} + \frac{1}{\eta_l}\,e^{-\xi}\sum_{m=1}^{\infty} \times \cos(\lambda_m\eta)\left[\frac{\sin(\xi\sqrt{(\lambda_m^2-1)})}{\sqrt{(\lambda_m^2-1)}} + \cos(\xi\sqrt{(\lambda_m^2-1)})\right] \quad (24)$$

$$\lim_{\Delta\eta\to 0}[Q(\eta,\xi)] = \frac{1}{\eta_l}\,e^{-\xi}\sum_{m=1}^{\infty} \times \lambda_m\sin(\lambda_m\eta)\frac{\sin(\xi\sqrt{(\lambda_m^2-1)})}{\sqrt{(\lambda_m^2-1)}}. \quad (25)$$

In as much as equations (24) and (25) represent solutions for an impulse energy source released at time $\xi = 0$ at location $\eta = 0$, given by

$$S(\eta,\xi) = \delta(\eta)\delta(\xi) \quad (26)$$

the temperature solution, equation (24), is related to the Green's function or fundamental solution, equation (17), by

$$\lim_{\Delta\eta\to 0}[\theta(\eta,\xi)] = \left[G(\eta,\xi|\eta_0,\xi_0) + \frac{1}{2}\frac{\partial G}{\partial\xi}\right]_{\eta_0=0,\xi_0=0}. \quad (27)$$

Also of interest is the limit as the finite region approaches a semi-infinite region, or the limit as $\eta_l \to \infty$. We take the limit of equations (21) and (23) to obtain

$$\lim_{\eta_l\to\infty}[\theta(\eta,\xi)] = \frac{e^{-\xi}}{\pi}\int_{\omega=0}^{\infty}\cos(\omega\eta) \times \frac{\sin(\omega\Delta\eta)}{\omega\Delta\eta}\left[\frac{\sin(\xi\sqrt{(\omega^2-1)})}{\sqrt{(\omega^2-1)}} + \cos(\xi\sqrt{(\omega^2-1)})\right]d\omega \quad (28)$$

$$\lim_{\eta_l\to\infty}[Q(\eta,\xi)] = \frac{e^{-\xi}}{\pi}\int_{\omega=0}^{\infty}\sin(\omega\eta) \times \frac{\sin(\omega\Delta\eta)}{\Delta\eta}\,\frac{\sin(\xi\sqrt{(\omega^2-1)})}{\sqrt{(\omega^2-1)}}\,d\omega. \quad (29)$$

The limiting forms, equations (28) and (29), should match solutions (21) and (23), respectively, any time before the pulsed energy source has reached the boundary at $\eta = \eta_l$, since energy will propagate as a wave exhibiting a distinct wave front with a finite propagation speed. Representative calculations showing the significance of the foregoing results are now presented.

RESULTS AND DISCUSSION

Numerical computations are performed in order to examine the behavior of a thermal pulse and the resulting temperature and heat flux distribution in a slab of thickness η_l, with both of its boundaries

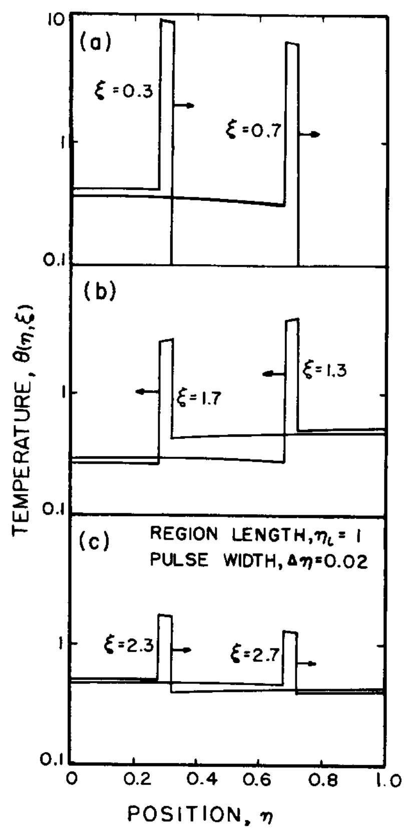

FIG. 2. Temperature distributions for a sequence of times resulting from a pulsed energy source.

insulated. The specific situation considered involves a pulsed energy source of width $\Delta\eta$ and concentration $1/\Delta\eta$, located in the medium adjacent to the boundary surface at $\eta = 0$ and releasing its energy spontaneously at time $\xi = 0$.

Equations (21) and (23) are utilized to compute the temperature and heat flux, respectively. We should mention that series solutions (21) and (23) are not in convenient form for numerical convergence, and require some manipulation using closed-form expressions, the details of which are quite lengthy and are not presented here.

Figure 2 shows a sequence of temperature plots as a function of position for a fixed pulse width of $\Delta\eta = 0.02$ and region length of $\eta_l = 1$. A striking feature shown in this figure is that severe temperatures persist in the form of a relatively concentrated thermal wave, which travels through the medium at a finite speed. In Fig. 2(a) the wave has not yet reached the opposite wall at $\eta = \eta_l$, thus there exists an undisturbed region ahead of the wave front. Since the wave has no knowledge of the wall at $\eta = \eta_l$ for the cases shown in Fig. 2(a), the solution behaves exactly the same as that for the semi-infinite region. Therefore, for times before the wave front reaches the opposite wall, the solution for the limiting case of $\eta_l \to \infty$, given by equation (28), and equation (21) yield identical results. We observe that a portion of the thermal energy is dissipated in the wake of the traveling wave front. Also, the wave front is distributed over a region of twice the original pulse width, $2\Delta\eta$, due to reflection from the insulated boundary surface at

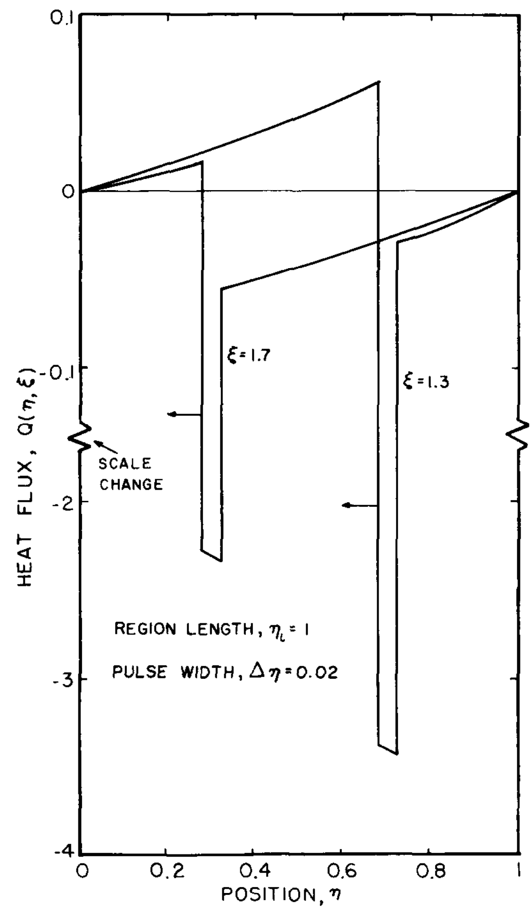

FIG. 3. Heat flux distributions corresponding to the conditions in Fig. 2(b).

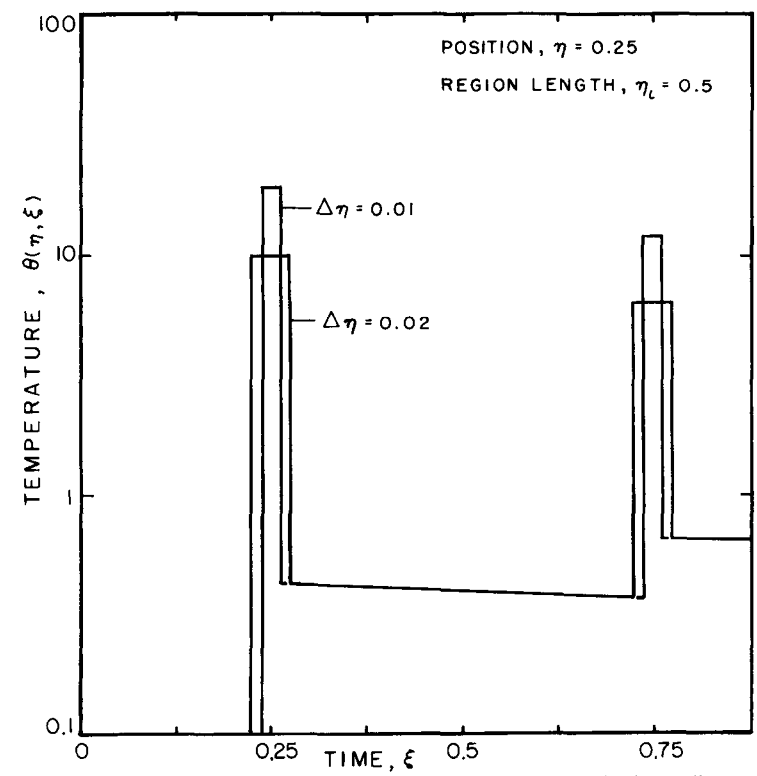

FIG. 4. Effect of pulse width on temperature at a given location in the medium.

$\eta = 0$, a phenomenon which will be discussed in detail with regard to Fig. 6.

Figure 2(b) shows the temperature distribution at times after the wave front has encountered the wall at $\eta = \eta_l$ and has been reflected. The wave reverses its direction of propagation at the wall and continues to dissipate energy in its wake, thus continuously building up the energy level of the region over which it passes. This pattern is continued in Fig. 2(c), where we observed the thermal wave after it has been reflected from the boundary at $\eta = 0$ and is making another pass through the region. The energy concentration in the wave front is exponentially diminishing since the original energy contained in the thermal pulse is dissipating in the wake of the traveling wave front, eventually spreading out evenly and resulting in a region of constant temperature $1/2\eta_l$, as can be seen from solution (21) for very large times.

Figure 3 shows the heat flux corresponding to the conditions shown in Fig. 2(b). Again a severe thermal wave front can be observed, where the heat flux is two orders of magnitude higher than anywhere else in the region. Since Fig. 3 represents a situation where the wave front is moving in the negative x-direction, the corresponding heat flux in the wave front and in the wake of the wave front is negative, while in the region ahead of the front, the flux is still positive. Thus, the hyperbolic heat equation predicts that a thermal disturbance tends to propagate in a given direction until its course is reversed by a barrier or wall.

The effect of the pulse width is demonstrated in Fig. 4, where the temperature field vs time at the center of a region of length $\eta_l = 0.5$ is shown. Decreasing the pulse width increases the energy concentration in the wave front with the peak becoming more severe. If we were to continue in the limit as the pulse concentration became infinite over an infinitesimally small pulse width, the limiting delta function behavior described by solutions (24) and (25) for the case $\Delta\eta \to 0$ would be obtained. Figure 4 also shows the influence of a finite wave speed in that for times less than $\eta - \Delta\eta$, the thermal disturbance has not yet been experienced at position η. Also, the amount of energy dissipated or left behind by the wave front is insensitive to the concentration of

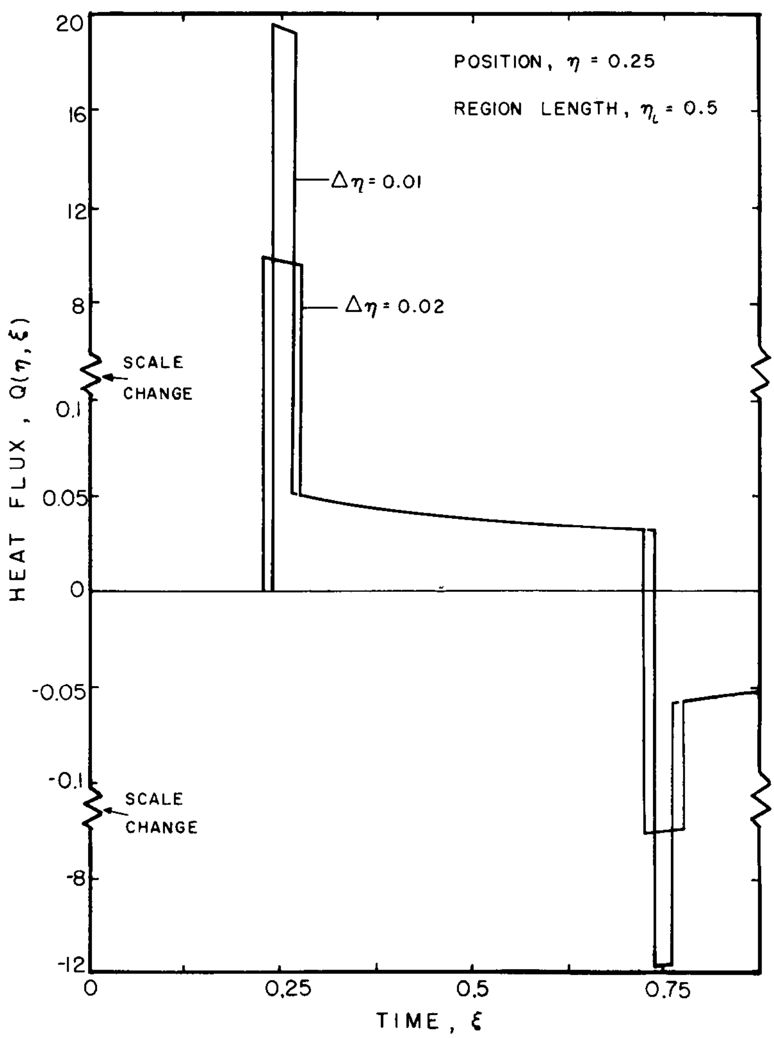

FIG. 5. Effect of pulse width on the heat flux for conditions corresponding to Fig. 4.

energy in the front, since alternating the concentration merely redistributes the energy and does not change the total energy originally released.

Next we examine Fig. 5, which shows the corresponding heat flux for the conditions shown in Fig. 4. Here again we observe a finite time required for a thermal disturbance to reach a given location. The first peaks in Fig. 5 represent the heat flux as the wave front passes for the first time in the positive x-direction, thus resulting in a relatively large position heat flux. However, the second peaks represent the behavior after the wave front has reflected from boundary surface $\eta = \eta_l$ and has returned while traveling in the negative x-direction, thus resulting in a relatively large negative heat flux. As with the temperature in Fig. 4, the heat flux is insensitive to the pulse concentration after the wave front has left the particular location.

Next we examine the details of the reflection of the thermal pulse from the insulated surfaces at $\eta = 0$ and η_l. Figure 6 shows a sequence of temperature plots vs position for three stages of reflection from boundary surface $\eta = 0$ (times $\xi = 0$, 0.025, and 0.05) and three stages of reflection from the boundary surface at $\eta = \eta_l = 1$ (times $\xi = 0.95, 0.975$, and 1). We follow the square pulse from the time it was instantaneously released at times $\xi = 0$, shown as a rectangle of height $\frac{1}{2}\Delta\eta = 10$, distributed over a region $\Delta\eta = 0.05$. At the instant that the pulse is generated, the energy contained in the pulse has no preferred direction of propagation and thus attempts to separate, with one-half the pulse moving in either direction. The portion moving in the positive x-direction encounters no difficulty and moves out into the region. However, the portion of the energy attempting to travel in the negative x-direction immediately encounters an impermeable wall and is reflected back into the medium. Thus at a time equal to one-half the pulse width ($\xi = \Delta\eta/2 = 0.025$), we observe a temperature distribution resembling two steps. At a time equal to the pulse width ($\xi = \Delta\eta = 0.05$), the portion of the original energy source which attempted to travel in the negative x-direction has been completely reflected from the boundary at $\eta = 0$ and a wave front of width $2\Delta\eta = 0.1$ moves out into the medium, as shown earlier in Fig. 2. Also included in Fig. 6 is the behavior of the thermal wave front after it has traversed the region and encountered the wall at $\eta = \eta_l = 1$. Here the behavior demonstrated during the initial reflection at $\eta = 0$ is reversed, with the wave front doubling its concentration due to reflection from the insulated surface before spreading out again to a width of $2\Delta\eta = 0.1$ and moving out into the region in the reverse direction. Of course the concentration in the wave front has been diminished as it reaches the wall at $\eta = \eta_l = 1$ due to dissipation during the propagation through the medium.

Figure 7 demonstrates the relative importance of the wave nature of energy transport by comparing a temperature vs time plot with the corresponding parabolic or diffusion solution. Here we are observing the temperature at the insulated wall at $\eta = \eta_l$ for a region length $\eta_l = 1$ and pulsed width $\Delta\eta = 0.02$. The parabolic solution predicts instantaneous energy diffusion at infinite propagation speed, thus the effect of a thermal disturbance adjacent to boundary $\eta = 0$ is immediately felt a finite distance away. The parabolic solution predicts that the initial energy pulse will diffuse evenly over the entire region after a time of

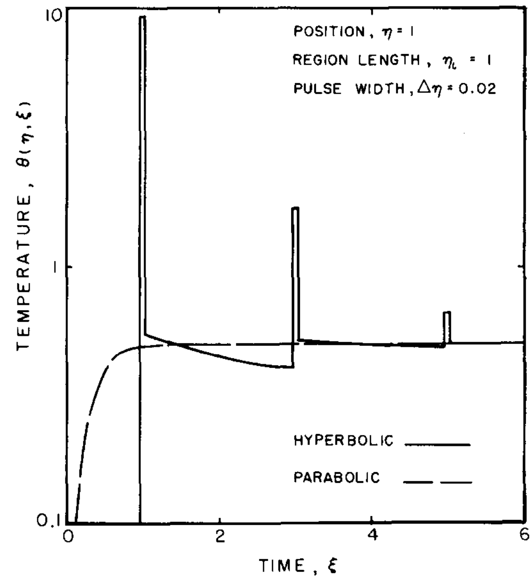

FIG. 7. Comparison of hyperbolic and parabolic temperature solutions resulting from a pulsed energy source.

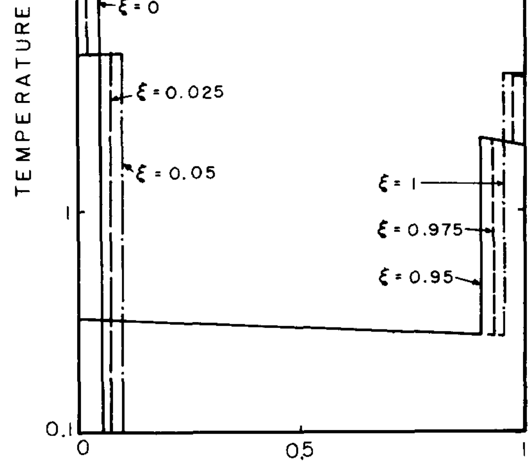

FIG. 6. Temperature distribution in the region at various times demonstrating the reflection of the thermal wave front.

approximately $\xi = 1$, while the hyperbolic solution shows that a severe concentration persists in the form of a concentrated wave front which reflects back and forth throughout the medium while gradually dissipating its energy.

In conclusion, we note that the wave nature of thermal energy transport is a phenomenon which is most often unrecognized or ignored. The results of the present analysis, showing the propagation and reflection of a severe concentration of thermal energy as predicted by the hyperbolic heat equation, are beyond the domain of the classical diffusion theory of heat conduction and can be prevailant under conditions of extremely short times or low temperatures.

Acknowledgement—This work was supported through the National Science Foundation grant No. MEA.8313301.

REFERENCES

1. V. Peshkov, 'Second sound' in helium II, *J. Phys., USSR* **VIII**, 381 (1944).
2. P. Vernotte, Les Paradoxes de la Théorie Continue de l'équation de la chaleur, *C. R.* **246**, 3154–3155 (1958).
3. M. Chester, Second sound in solids, *Phys. Rev.* **131**(5), 2013–2015 (1963).
4. B. Bertman and D. J. Sandiford, 'Second sound' in solid helium, *Scient. Am.* **222**(5), 92–101 (1970).
5. S. Goldstein, On diffusion by discontinuous movements, and on the telegraph equation, *Q. Jl Mech. Appl. Math.* **IV**(2), 129–156 (1951).
6. H. D. Weymann, Finite speed of propagation in heat conduction, diffusion, and viscous shear motion, *Am. J. Phys.* **35**(6), 488–496 (1967).
7. M. E. Gurtin and A. C. Pipkin, A general theory of heat conduction with finite wave speeds, *Archs Ration. Mech. Analysis* **31**, 113–126 (1968).
8. P. J. Chen, On the growth and decay of temperature rate waves of arbitrary form, *Z. Angew. Math. Mech.* **20**, 448–453 (1969).
9. D. E. Amos and P. J. Chen, Transient heat conduction with finite wave speeds, *J. Appl. Mech.* **37**, 1145–1146 (1970).
10. K. J. Baumeister and T. D. Hamill, Hyperbolic heat-conduction equation—a solution for the semi-infinite body problem, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **91**, 543–548 (1969).
11. Y. Taitel, On the parabolic, hyperbolic and discrete formulation of the heat conduction equation, *Int. J. Heat Mass Transfer* **15**, 369–371 (1972).
12. D. C. Wiggert, Analysis of early-time transient heat conduction by method of characteristics, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **99**, 35–40 (1977).
13. B. Vick and M. N. Özişik, Growth and decay of a thermal pulse predicted by the hyperbolic heat conduction equation, *Trans. Am. Soc. Mech. Engrs*, Series C, *J. Heat Transfer* **105**, 902–907 (1983).
14. J. C. Maxwell, On the dynamical theory of gases, *Phil. Trans. R. Soc.* **157**, 49–88 (1867).
15. J. F. Gibbons, L. D. Hess and T. W. Sigmon (editors), *Laser and Electron-beam Solid Interactions and Material Processing*. Elsevier North Holland, New York (1981).
16. B. R. Appleton and G. K. Cellar (editors), *Laser and Electron-beam Interactions with Solids*, Elsevier Science, New York (1982).
17. L. D. Hess, S. A. Kokorowski, G. L. Olson and G. Yaron, Laser processing of silicon for advanced microelectronic devices and circuits, in ref. [15], pp. 307–319 (1981).
18. N. Bloembergen, H. Kurz, J. M. Liu and R. Yen, Fundamentals of energy transfer during picosecond irradiation of silicon, in ref. [16], pp. 3–11 (1982).
19. B. Stritzker, A. Pospieszczyk and J. A. Tagle, Measurement of lattice temperature of silicon during pulsed laser annealing, *Phys. Rev. Lett.* **47**(5), 356–358 (1981).
20. P. Baeri, S. U. Campisano, G. Foti and E. Rimini, A melting model for pulsing-laser annealing of implanted semiconductors, *J. Appl. Phys.* **50**(2), 788–797 (1979).
21. W. E. Maher and R. B. Hall, Pulsed laser heating profile width and changes in total coupling with pulse length and pressure, *J. Appl. Phys.* **51**(3), 1338–1344 (1980).
22. B. C. Larson, C. W. White and T. S. Noggle, Synchrotron X-ray diffraction study of silicon during pulsed-laser annealing, *Phys. Rev. Lett.* **48**(5), 337–340 (1982).
23. S. H. Chan, M. J. D. Low and W. K. Mueller, Hyperbolic heat conduction in catalytic supported crystallites, *A.I.Ch.E. Jl* **17**(6), 1499–1501 (1971).
24. L. Spenadel and M. Boudart, Dispersion of platinum on supported catalysts, *J. Phys. Chem.* **64**, 204–207 (1960).
25. S. F. Alder and J. J. Keavney, The physical nature of supported platinum, *J. Phys. Chem.* **64**, 208–212 (1960).
26. J. A. Cusumano and M. J. D. Low, Infrared radiometry of thermal transients on surfaces, *J. Catalysis* **17**, 98–105 (1970).
27. D. Luss, Temperature rise of catalytic supported crystallites, *Chem. Engng J.* **1**, 311–316 (1970).
28. P. M. Morse and H. Feshback, *Methods of Theoretical Physics*, Part I, pp. 865–869. McGraw-Hill, New York (1953).
29. M. N. Özişik, *Heat Conduction*. Wiley, New York (1980).

PROPAGATION ET REFLEXION D'ONDES THERMIQUES DANS UN MILIEU FINI

Résumé—Pour des situations qui concernent des temps extrêmement courts après le début du chauffage ou pour des températures proches du zéro absolu, la théorie classique de la diffusion de la chaleur n'est plus valable car la nature ondulatoire du transfert de chaleur devient dominant. Des solutions analytiques sont développées pour l'équation hyperbolique qui décrit la nature ondulatoire du transfert thermique dans une plaque finie avec des frontières isolées et soumise à une source volumétrique d'énergie dans le milieu. Les solutions analytiques exactes développées pour le champ de température et le flux thermique montre que l'évolution d'une impulsion concentrée d'énergie donne lieu à un front thermique important qui se déplace à travers le milieu avec une vitesse finie, en dissipant l'énergie dans son passage et qui se réfléchit sur les surfaces isolées.

FORTPFLANZUNG UND REFLEXION VON THERMISCHEN WELLEN IN EINEM BEGRENZTEN MEDIUM

Zusammenfassung—Im Falle extrem kurzer Betrachtungszeiträume unmittelbar nach Beginn eines instationären Wärmeleitvorganges oder für sehr tiefe Temperaturen nahe dem absoluten Nullpunkt verliert die klassische Theorie der Wärmeleitung ihre Gültigkeit, da dann die Wellennatur des thermischen Energietransports überwiegt. In dieser Arbeit werden analytische Lösungen für die hyperbolische Wärmeleitungsgleichung entwickelt, die den thermischen Energietransport entsprechend seiner Wellennatur in einer endlichen Platte mit isolierten Rändern unter dem Einfluß einer inneren volumetrischen Wärmequelle beschreibt. Die exakten analytischen Lösungen für das Temperaturfeld und den Wärmestrom zeigen, daß ein konzentrierter Energieimpuls eine ausgeprägte thermische Wellenfront zur Folge hat, die sich mit endlicher Geschwindigkeit durch das Medium bewegt, in ihrem Nachlauf Energie dissipiert und an den isolierten Begrenzungsflächen reflektiert wird.

РАСПРОСТРАНЕНИЕ И ОТРАЖЕНИЕ ТЕПЛОВЫХ ВОЛН В СРЕДЕ КОНЕЧНЫХ РАЗМЕРОВ

Аннотация—На очень коротких временах, следующих за началом нестационарного процесса, или при очень низких температурах, близких к абсолютному нулю, классическая теория теплопроводности становится непригодной, поскольку доминирующей становится волновая природа переноса тепловой энергии. Предложены аналитические решения гиперболического уравнения теплопроводности, учитывающего волновую природу переноса тепловой энергии в плите конечных размеров с теплоизолированными стенками и объемным источником энергии. Точные аналитические решения, предложенные для поля температур и теплового потока, показывают, что инициирование сосредоточенного импульса энергии индуцирует мощный фронт тепловой волны, который распространяется через среду с конечной скоростью, рассеивая энергию в следе и отражаясь от теплоизолированных поверхностей.